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Normalized matching property of the subgroup lattice of an abelian p-group ☆

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Abstract

Let $L_{(k^n)}(p)$ denote the subgroup lattice of the abelian p-group

$(\mathbb{Z}/p^k\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^k\mathbb{Z}) \quad (n \text{ times}).$

In a previous paper (Ann. of Combin. 2 (1998) 85), we proved that $L_{(k^n)}(p)$ has the Sperner property. In this paper, we prove that for any positive integers n and k, there is a positive integer $N(n,k)$ such that $L_{(k^n)}(p)$ has the normalized matching property when $p > N(n,k)$. As a consequence, $L_{(k^n)}(p)$ has the strong Sperner property, LYM property and it is a symmetric chain order when p is sufficiently large.
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1. Introduction

In 1928 Sperner [21] showed that if $\mathscr{F}$ is a collection of subsets of $\{1,2,\ldots,n\}$ such that no set in $\mathcal{F}$ contains others, then

$$|\mathscr{F}| \leqslant \binom{n}{\lceil n/2 \rceil}.$$

Since then there appeared many refinements and generalizations of Sperner's theorem. In 1967, Rota posed a famous research problem in [18]: Prove or disprove the analogue,

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for the lattice of partitions of an n-element set, of Sperner's Theorem. Although the problem was shown to be negative for sufficiently large n by Canfield [6] in 1978, it drew attention to posets, see e.g. [9,10,12,13,16,19,22], etc.

A finite poset P is graded if every maximal chain of P has the same length, denoted by $r(P)$ which is called the rank of P. Thus, for each $x \in P$ every maximal chain with x as the top element has the same length, denoted by $r(x)$. Here the length of a chain with k elements is $k-1$. Let P_i denote the ith rank of P which consists of all $x \in P$ with $r(x)=i$, $i=0,1,\dots,r(P)$. A graded poset P with rank n is rank symmetric if $|P_i|=|P_{n-i}|$ for $0 \leqslant i \leqslant n/2$. It is rank unimodal if $|P_0| \leqslant |P_1| \leqslant \cdots \leqslant |P_k| \geqslant |P_{k+1}| \geqslant \cdots \geqslant |P_n|$ for some $0 \leqslant k \leqslant n$.

An antichain is a subset $A \subseteq P$ of which no two elements are comparable in P. A k-family is a subset of P that contains no chains of length k. Thus a 1-family is just an antichain. Clearly, each rank P_i of P is an antichain, and each union of k ranks of P is a k-family.

We say a graded poset P is k-Sperner, or has the k-Sperner property, if a union of k ranks of P is a maximal-sized k-family in P. 1-Sperner is just called Sperner. We say P is strong Sperner, or has the strong Sperner property if it is k-Sperner for all k. A symmetric chain order is a graded poset that can be partitioned into chains, the ranks of each of whose elements are consecutive and symmetric about the middle rank.

For $A \subseteq P_i$, define $\nabla(A)=\{b \in P_{i+1}\colon\ b \geqslant a \text{ for some } a \in A\}$. We say P has the normalized matching property (NM), or is a normal poset if

$$\frac{|\nabla(A)|}{|A|} \geqslant \frac{|P_{i+1}|}{|P_i|} \quad \text{for all } A \subseteq P_i,\ i=0,1,\dots,r(P)-1. \tag{1}$$

We say P has the LYM property, or is a LYM poset if for every antichain A in P, the following LYM inequality holds:

$$\sum_{i=0}^{r(P)} \frac{|A \cap P_i|}{|P_i|} \leqslant 1.$$

It is well known that a symmetric chain order is strong Sperner [12], and the strong Sperner property is also derived from the LYM property; the NM property is equivalent to the LYM property [14]; and a normal or a LYM poset is a symmetric chain order if it is rank unimodal and rank symmetric [11]. For more details on these relations, we refer to [7,13].

Many posets have all or some of the properties, for example, the boolean lattice B_n of subsets of an n-set and the linear lattice $L_n(q)$ of subspaces of an n-dimensional vector space over $GF(q)$ have all the properties, the lattice $L(n,m)$ of order ideals of a product of an n-chain and an m-chain is strong Sperner, but it is still open whether or not it is a symmetric chain order as asked by Stanley [22] (see also [2,7]).

In this paper, we investigate the normalized matching property of the lattice $L_{(k^n)}(p)$ of subgroups of the abelian p-group

$$(\mathbb{Z}/p^k\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p^k\mathbb{Z}) \quad (n \text{ times}).$$

It is well known that $L_{(k^n)}(p)$ is rank unimodal and rank symmetric. Therefore, that it has the normalized matching property would imply that it has all other properties.

There are at least two reasons to consider this lattice important in combinatorics: it is a generalization of $L_n(q)$, and a p-analog of the lattice of submultisets of the multiset $\{1^k, 2^k, \ldots, n^k\}$, where i^k denotes a sequence of k i's (cf. [5,20]). So $L_{(k^n)}(p)$ should share the same Sperner properties as those lattices, it has been conjectured for a long time that this lattice is Sperner, strong Sperner, even a symmetric chain order (cf. [4,7, p. 251, 24]). However, all these properties were established only for $k=1$. In [24] Stanley proved that $L_{(2^n)}(p)$ is Sperner (see also [7, pp. 251–253]), and in [25], we confirmed the Sperner property for all n and k. Our argument was based on a theorem, due to Kleitman et al. [15] (see also [1, p. 400], [7, p. 160, 8]).

KEL Theorem. *Let P be a finite poset and Γ a permutation group acting on P which preserves the order relation on P. Then P contains a maximal sized antichain which is invariant under the action of Γ, in other words, which is a union of Γ-orbits.*

The generalization of the KEL Theorem to k-families was presented in [10] (see also [7, p. 160]). By this result we proved in [26] that $L_{(k^n)}(p)$ has the strong Sperner property when p is sufficiently large.

To apply this approach for studying the NM property of a poset, we first present a NM version of the KEL Theorem as follows.

Theorem 1. *Let P and Γ be the same as in the KEL Theorem. Then for every i, there is a subset A_i of P_i which is invariant under the action of Γ and*

$$\frac{|\nabla(A_i)|}{|A_i|} = \min\left\{\frac{|\nabla(B)|}{|B|}\colon\ B \subseteq P_i\right\}. \tag{2}$$

This theorem implies that P has the NM property if (1) holds for all A's that are invariant under the action of Γ. By Theorem 1, we obtain the following theorem.

Theorem 2. *Let n and k be positive integers and p a prime number. Then there is a positive integer $N(n,k)$ such that $L_{(k^n)}(p)$ has the normalized matching property when $p > N(n,k)$.*

As a consequence of Theorem 2, we have

Corollary 3. *$L_{(k^n)}(p)$ has the strong Sperner property, LYM property and is a symmetric chain order when p is sufficiently large.*

We conjecture that the theorem holds for any small prime p. But it is too complicated to show in the same way.

We shall, in Section 2, prove a more general theorem than Theorem 1 and illustrate our approach; in Section 3, we establish some lemmas which are needed for the proof of Theorem 2, while the proof will be completed in Section 4.

2. Weighted posets and quotient posets

As stated in [7], some extremal problems can be considered in a weighted poset (P,w), which is a poset P together with a function (called a weight function) w from P into the set of non-negative real numbers. The weight $w(A)$ of a subset A of P is defined by $w(A)=\sum_{a\in A} w(a)$. Every poset P can be considered as a weighted poset (P,w), where $w\equiv 1$, that is, $w(x)=1$ for all $x\in P$.

All the properties introduced in the last section can be defined for a weighted poset. For example, a weighted poset (P,w) has the NM property if

$$\frac{w(\nabla(A))}{w(A)} \geqslant \frac{w(P_{i+1})}{w(P_i)} \quad \text{for all } A\subseteq P_i,\ i=0,1,\dots,r(P)-1. \tag{3}$$

Now, let Γ be a permutation group on P which preserves the order relation and the weight on P, that is, for every $\gamma\in\Gamma$, $x\leqslant y$ in P implies $\gamma(x)\leqslant\gamma(y)$, and $w(\gamma(x))=w(x)$. Then we have the quotient poset $(P/\Gamma, w_\Gamma)$, where P/Γ consists of the Γ-orbits ordered as follows: $\Gamma x\leqslant\Gamma y$ in P/Γ if $x'\leqslant y'$ for some $x'\in\Gamma x$ and $y'\in\Gamma y$, and the weight function w_Γ is given by $w_\Gamma(\Gamma x)=|\Gamma x|w(x)$. The weighted KEL Theorem (see [7, p. 160, Theorem 4.5.5]) says that (P,w) has the k-Sperner property if and only if $(P/\Gamma, w_\Gamma)$ does. The following theorem is a little more general than Theorem 1.

Theorem 4. *(P,w) has the NM property if and only if $(P/\Gamma, w_\Gamma)$ does.*

Proof. Let A be a subset of P_i, $0\leqslant i\leqslant r(P)$. Set $\Gamma A=\{\gamma(a)\colon\ a\in A \text{ and } \gamma\in\Gamma\}$. From definition we see that $w_\Gamma(\Gamma A)=w(\Gamma A)\geqslant w(A)$, and the equality holds if and only if A is Γ-invariant, in other words, A is a union of Γ-orbits. We thus have

$$\begin{aligned}\min\left\{\frac{w(\nabla(A))}{w(A)}\colon\ A\subseteq P_i\right\} &\leqslant \min\left\{\frac{w(\nabla(\Gamma B))}{w(\Gamma B)}\colon\ B\subseteq P_i\right\}\\ &= \min\left\{\frac{w_\Gamma(\nabla(\Gamma B))}{w_\Gamma(\Gamma B)}\colon\ B\subseteq P_i\right\}.\end{aligned}$$

Therefore, the NM property of (P,w) implies the NM property of $(P/\Gamma, w_\Gamma)$.

Conversely, suppose $(P/\Gamma, w_\Gamma)$ has the NM property, that is, for every $B\subseteq P_i$,

$$\frac{w_\Gamma(\nabla(\Gamma B))}{w_\Gamma(\Gamma B)} \geqslant \frac{w_\Gamma(\nabla(\Gamma P_i))}{w_\Gamma(\Gamma P_i)} = \frac{w(\nabla(P_i))}{w(P_i)}.$$

Clearly, in order to confirm the NM property of (P,w), it suffices to show that there is a Γ-invariant subset A_i of P_i satisfying

$$\frac{w(\nabla(A_i))}{w(A_i)} = \min\left\{\frac{w(\nabla(B))}{w(B)}\colon\ B\subseteq P_i\right\}. \tag{4}$$

Suppose A is a minimal subset of P_i such that (4) holds. If $\Gamma A \neq A$, then there is a $\gamma \in \Gamma$ such that $B = \gamma(A) \neq A$. In this case, if $B \cap A \neq \emptyset$, then

$$\frac{w(\nabla(A))}{w(A)} \leqslant \frac{w(\nabla(A \cup B))}{w(A \cup B)} = \frac{w(\nabla(A) \cup \nabla(B))}{w(A \cup B)}$$
$$= \frac{2w(\nabla(A)) - w(\nabla(A) \cap \nabla(B))}{2w(A) - w(A \cap B)} \leqslant \frac{2w(\nabla(A)) - w(\nabla(A \cap B))}{2w(A) - w(A \cap B)},$$

which implies

$$\frac{w(\nabla(A))}{w(A)} \geqslant \frac{w(\nabla(A \cap B))}{w(A \cap B)},$$

a contradiction. This shows that $\gamma(A) = A$ or $\gamma(A) \cap A = \emptyset$ for all $\gamma \in \Gamma$. If the former, then A is Γ-invariant, and the proof is finished. If the latter, letting $\Gamma A = \{A = \gamma_1(A), \gamma_2(A), \ldots, \gamma_r(A)\}$, then $w(\Gamma A) = rw(A)$. In this case we see that $w(\nabla(\Gamma A)) \leqslant rw(\nabla(A))$, and

$$\frac{w(\nabla(\Gamma A))}{w(\Gamma A)} \leqslant \frac{w(\nabla(A))}{w(A)}.$$

Thus, we can take ΓA as A_i. □

Remark 5. Konrad Engel told the author a more general result than Theorem 4: If there is a flow morphism ϕ from (P, w) onto (Q, v) then (P, w) has the normalized matching property if and only if (Q, v) does. The proof follows by using regular coverings (see [7, p. 150, Theorem 4.5.1]).

For our purpose, let $P = L_{(k^n)}(p)$, $\Gamma = GL_n(\mathbb{Z}/p^k\mathbb{Z})$, the general linear group over $\mathbb{Z}/p^k\mathbb{Z}$ of dimension n, and $L(n,k)$ the set of all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant k$, where λ_i is called the ith part of λ.

It is well known that for any $x \in P$ there is a unique $\lambda \in L(n,k)$ such that Γx is the set of all subgroups of type λ in P. Therefore, the quotient poset P/Γ is isomorphic to $L(n,k)$. As the quotient poset, $L(n,k)$ may be ordered as follows:

$$\mu \leqslant \lambda \text{ in } L(n,k) \text{ if and only if } \mu_i \leqslant \lambda_i \text{ holds for each } i = 1, 2, \ldots, n.$$

With respect to this order, $L = L(n,k)$ is a graded lattice. By L_u we denote the uth rank of L, i.e., L_u consists of all $\lambda \in L$ such that $\sum_{i=1}^n \lambda_i = u$ $(0 \leqslant u \leqslant nk)$.

Thus, the NM property of the lattice $L_{(k^n)}(p)$ is obtained by that of the weighted lattice $(L(n,k), w)$, where $w(\lambda)$ is the number of subgroups of type $\lambda \in L(n,k)$ in $L_{(k^n)}(p)$. For convenience, let $f_\lambda(p)$ denote the weight $w(\lambda)$ of $\lambda \in L(n,k)$ and let $|A|_p$ denote the weight of a subset A of $L(n,k)$, that is,

$$|A|_p = \sum_{\lambda \in A} f_\lambda(p).$$

By Theorem 1, $L_{(k^n)}(p)$ has the NM property if and only if

$$\frac{|\nabla(A)|_p}{|A|_p} \geqslant \frac{|L_{u+1}|_p}{|L_u|_p} \quad \text{for all } u=0,1,\ldots,nk-1 \text{ and for each } A \subseteq L_u.$$

For $\lambda \in L(n,k)$, denote $m(\lambda)=(m_0,m_1,\ldots,m_k)$, where $m_i=m_i(\lambda)=|\{j\colon\ \lambda_j=i\}|$. By definition we have the following relations immediately:

$$\begin{aligned} &n=m_0+m_1+\cdots+m_k,\\ &\lambda_i'=m_i+\cdots+m_k,\ 1\leqslant i\leqslant k, \end{aligned} \tag{5}$$

where $\lambda_i'=|\{j\colon\ \lambda_j\geqslant i\}|$, or combinatorially, λ' is the conjugate partition of λ.

It is well known that (see, e.g. [3] or [4])

$$f_\lambda(p)=\prod_{i\geqslant 1} p^{\lambda'_{i+1}(n-\lambda'_i)} \begin{bmatrix} n-\lambda'_{i+1} \\ \lambda'_i-\lambda'_{i+1} \end{bmatrix}_p, \tag{6}$$

where

$$\begin{bmatrix} n \\ a \end{bmatrix}_p = \begin{bmatrix} n \\ a,n-a \end{bmatrix}_p$$

is the p-binomial coefficient. Generally, if $(a_1,a_2,\ldots,a_k)$ is a sequence of non-negative integers summing to n, we may define the p-multinomial coefficient

$$\begin{bmatrix} n \\ a_1,a_2,\ldots,a_k \end{bmatrix}_p = \frac{\langle n\rangle!}{\langle a_1\rangle!\langle a_2\rangle!\cdots\langle a_k\rangle!},$$

where $\langle i\rangle!=\langle 1\rangle\langle 2\rangle\cdots\langle i\rangle$ and $\langle j\rangle=1+p+\cdots+p^{j-1}$. It is straightforward to verify [23, p. 26] that

$$\begin{bmatrix} n \\ a_1,a_2,\ldots,a_k \end{bmatrix}_p = \begin{bmatrix} n \\ a_1 \end{bmatrix}_p \begin{bmatrix} n-a_1 \\ a_2 \end{bmatrix}_p \begin{bmatrix} n-a_1-a_2 \\ a_3 \end{bmatrix}_p \cdots \begin{bmatrix} a_k \\ a_k \end{bmatrix}_p,$$

from which we may rewrite the expression of $f_\lambda(p)$ by (6) in a form involving the p-multinomial coefficients, as stated in Lemma 2.1 in [25]:

$$f_\lambda(p)=\begin{bmatrix} n \\ m_0,m_1,\ldots,m_k \end{bmatrix}_p p^{F(m)},$$

where

$$F(m)=\sum_{0\leqslant i<j\leqslant k}(j-i-1)m_im_j.$$

Note that $f_\lambda(p)$ is a polynomial in p, and so is $|A|_p$ for each $A\subseteq L$. By $\deg f(p)$ we denote the degree of the polynomial $f(p)$, and define $\deg f(p)/g(p)=\deg f(p)-\deg g(p)$. In order to compare the absolute values of polynomials $f(p)$ and $g(p)$, we consider their degrees $\deg f(p)$ and $\deg g(p)$. If $\deg f(p)>\deg g(p)$, then $|f(p)|>|g$

$(p)|$ when p is sufficiently large. With this condition on p in Theorem 2, we shall follow such an argument: if the fraction $f(p)/g(p)$ is positive and $\deg f(p)/g(p) > \deg h(p)/l(p)$, then $f(p)/g(p) > h(p)/l(p)$.

3. Lemmas

For any real number x, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the greatest integer not greater than x and the least integer not less than x, respectively.

For any $\lambda \in L(n,k)$, λ_i always denotes the ith part of λ, that is, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant k$. Generally, for any n-tuple of non-negative integers, say $X = (x_1, x_2, \ldots, x_n)$, where $0 \leqslant x_i \leqslant k$, we let $X_{\leqslant}$ denote the corresponding partition that belongs to $L(n,k)$.

For the order $\leqslant$ on $L(n,k)$, we define an order-raising operator ϕ_t $(1 \leqslant t \leqslant n)$ which maps some elements of L_u into L_{u+1} as follows:

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in L_u$ with $\lambda_t = i < k$ and $m(\lambda) = (m_0, m_1, \ldots, m_k)$. Then

$$\phi_t(\lambda) = (\lambda_1, \ldots, \lambda_t + 1, \ldots, \lambda_n)_{\leqslant}$$

or

$$m(\phi_t(\lambda)) = (\ldots, m_i - 1, m_{i+1} + 1, \ldots). \tag{7}$$

Now, we introduce another partial order "$\preccurlyeq$" on L which is called natural ordering in [17, p. 7]:

$$\lambda \preccurlyeq v \text{ iff } \lambda_1 + \cdots + \lambda_i \geqslant v_1 + \cdots + v_i, \quad i = 1, 2, \ldots, n. \tag{8}$$

It is well known that for any $\lambda, v \in L_u$, $\lambda \preccurlyeq v$ if and only if v is obtained from λ by a series of the transformations of the form

$$\gamma_{s,t}(\lambda) = (\ldots, \lambda_s - 1, \ldots, \lambda_t + 1, \ldots)_{\leqslant},$$

where $1 \leqslant s < t \leqslant n$. Let $\lambda_s = i$ and $\lambda_t = j$. Then

$$m(\gamma_{s,t}(\lambda)) = \begin{cases} (\ldots, m_{i-1} + 1, m_i - 1, \ldots, m_j - 1, m_{j+1} + 1, \ldots) & \text{if } i < j, \\ (\ldots, m_{i-1} + 1, m_i - 2, m_{i+1} + 1, \ldots) & \text{if } i = j. \end{cases} \tag{9}$$

Lemma 6. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in L_u$ with $m(\lambda) = (m_0, m_1, \ldots, m_k)$.*
(i) *For any index t with $1 \leqslant t \leqslant n$ and $\lambda_t = i < k$ we have*

$$\frac{f_{\phi_t(\lambda)}(p)}{f_\lambda(p)} = \frac{\langle m_i \rangle}{\langle m_{i+1} + 1 \rangle} p^{\sum_{l<i} m_l - \sum_{h>i+1} m_h}. \tag{10}$$

(ii) *For any indices s and t with* $s<t$, $\lambda_s=i$ *and* $\lambda_t=j$, *we have*

$$\frac{f_{\gamma_{s,t}(\lambda)}(p)}{f_\lambda(p)}=\begin{cases}\dfrac{\langle m_i\rangle\langle m_j\rangle}{\langle m_{i-1}+1\rangle\langle m_{j+1}+1\rangle}\,p^{m_{i-1}+m_i+2(m_{i+1}+\cdots+m_{j-1})+m_j+m_{j+1}} & \text{if } i<j,\\[2mm] \dfrac{\langle m_i\rangle\langle m_i-1\rangle}{\langle m_{i-1}+1\rangle\langle m_{i+1}+1\rangle}\,p^{1+m_{i-1}+m_{i+1}} & \text{if } i=j.\end{cases} \tag{11}$$

Proof. Write

$$m(\phi_t(\lambda))=m'=(m'_0,m'_1,\ldots,m'_k)$$

and

$$m(\gamma_{s,t}(\lambda))=m''=(m''_0,m''_1,\ldots,m''_k).$$

Then m' and m'' can be expressed in m, as indicated in (7) and (9). From this we have

$$\frac{f_{\phi_t(\lambda)}(p)}{f_\lambda(p)}=\frac{\langle m_0\rangle!\langle m_1\rangle!\cdots\langle m_k\rangle!}{\langle m'_0\rangle!\langle m'_1\rangle!\cdots\langle m'_k\rangle!}\,p^{F(m')-F(m)}=\frac{\langle m_i\rangle}{\langle m_{i+1}+1\rangle}\,p^{F(m')-F(m)}$$

and

$$\frac{f_{\gamma_{s,t}(\lambda)}(p)}{f_\lambda(p)}=\frac{\langle m_0\rangle!\langle m_1\rangle!\cdots\langle m_k\rangle!}{\langle m''_0\rangle!\langle m''_1\rangle!\cdots\langle m''_k\rangle!}\,p^{F(m'')-F(m)}$$

$$=\begin{cases}\dfrac{\langle m_i\rangle\langle m_j\rangle}{\langle m_{i-1}+1\rangle\langle m_{j+1}+1\rangle}\,p^{F(m'')-F(m)} & \text{if } i<j,\\[2mm] \dfrac{\langle m_i\rangle\langle m_i-1\rangle}{\langle m_{i-1}+1\rangle\langle m_{i+1}+1\rangle}\,p^{F(m'')-F(m)} & \text{if } i=j.\end{cases}$$

Let $F(X)=\sum_{0\leqslant l<h\leqslant k}(h-l-1)x_lx_h$ be a polynomial in $X=(x_0,x_1,\ldots,x_k)$. Then for any $\varepsilon=(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_k)$ we have $F(X+\varepsilon)-F(X)=F(\varepsilon)+\sum_{0\leqslant l<h\leqslant k}(h-l-1)(\varepsilon_lx_h+x_l\varepsilon_h)$.

(a) Put $\varepsilon_i=-1$, $\varepsilon_{i+1}=1$ and $\varepsilon_l=0$ if $l\neq i,\ i+1$. Then $m'=m+\varepsilon$. In this case, it is easy to verify that $F(\varepsilon)=0$ and

$$\begin{aligned}F(m')-F(m)&=\sum_{l<i}(i-l-1)m_l\varepsilon_i+\sum_{l<i+1}(i-l)m_l\varepsilon_{i+1}\\&\quad+\sum_{i<h}(h-i-1)\varepsilon_im_h+\sum_{i+1<h}(h-i-2)\varepsilon_{i+1}m_h\\&=\sum_{l<i}m_l-\sum_{h>i+1}m_h,\end{aligned}$$

which yields (10).

(b) Put $\varepsilon_{i-1}=1$, $\varepsilon_i=-1$, $\varepsilon_j=-1$, $\varepsilon_{j+1}=1$ and $\varepsilon_l=0$ if $l\neq i-1,i,j,j+1$. Then $m''=m+\varepsilon$. In this case, we have that $F(\varepsilon)=(j-i)\varepsilon_{i-1}\varepsilon_j+(j-i-1)\varepsilon_i\varepsilon_j+(j-i+1)$

$\varepsilon_{i-1}\varepsilon_{j+1} + (j-i)\varepsilon_i\varepsilon_{j+1} = -(j-i) + (j-i-1) + (j-i+1) - (j-i) = 0$ and

$$\begin{aligned} F(m'') - F(m) = &\sum_{l<i-1} (i-l-2)m_l + \sum_{h>i-1} (h-i)m_h - \sum_{l<i} (i-l-1)m_l \\ &- \sum_{h>i} (h-i-1)m_h - \sum_{l<j} (j-l-1)m_l - \sum_{h>j} (h-j-1)m_h \\ &+ \sum_{l<j+1} (j-l)m_l + \sum_{h>j+1} (h-j-2)m_h = -\sum_{l<i-1} m_l \\ &+ \sum_{h>i} m_h + \sum_{l<j} m_l - \sum_{h>j+1} m_h \\ = &\sum_{i-1\leqslant l\leqslant j-1} m_l + \sum_{i+1\leqslant h\leqslant j+1} m_h \\ = &\, m_{i-1} + m_i + 2(m_{i+1} + \cdots + m_{j-1}) + m_j + m_{j+1}, \end{aligned}$$

which yields the first equality in (11).

(c) Put $\varepsilon_{i-1} = 1$, $\varepsilon_i = -2$, $\varepsilon_{i+1} = 1$ and $\varepsilon_l = 0$ if $l \neq i-1, i, i+1$. Then $m'' = m + \varepsilon$. In this case, it is clear that $F(\varepsilon) = \varepsilon_{i-1}\varepsilon_{i+1} = 1$ and

$$\begin{aligned} F(m'') - F(m) = 1 &+ \sum_{l<i-1} (i-l-2)m_l + \sum_{h>i-1} (h-i)\, m_h - 2\sum_{l<i} (i-l-1)m_l \\ &- 2\sum_{h>i} (h-i-1)m_h + \sum_{l<i+1} (i-l)m_l \\ &+ \sum_{h>i+1} (h-i-2)m_h = 1 + m_{i-1} + m_{i+1}, \end{aligned}$$

which yields the second equality in (11). □

From (10) we see that

$$\deg \frac{f_{\phi_t(\lambda)}(p)}{f_\lambda(p)} = n - 1 - 2(m_{i+1}(\lambda) + \cdots + m_k(\lambda)), \quad \text{where } i = \lambda_t. \tag{12}$$

In particular, we define $\phi(\lambda) = \phi_{t_0}(\lambda)$, where $t_0 = n - m_k(\lambda)$, the greatest index t such that $\lambda_t < k$. Then, by (12), we have

$$\deg \frac{f_{\phi(\lambda)}(p)}{f_\lambda(p)} = n - 1 - 2m_k(\lambda). \tag{13}$$

From (11) we see that

$$\deg \frac{f_{\gamma_{s,t}(\lambda)}(p)}{f_\lambda(p)} = \begin{cases} 2(m_i(\lambda) + \cdots + m_j(\lambda)) - 2 & \text{if } \lambda_s = i < \lambda_t = j, \\ 2(m_i(\lambda) - 1) & \text{if } \lambda_s = \lambda_t = i \end{cases} \tag{14}$$

by which it is easy to verify that

$$\deg f_\mu(p) \geqslant \deg f_\lambda(p) + 2 \tag{15}$$

holds if λ and μ belong to the same rank of L and $\lambda \prec \mu$.

Lemma 7. *Let $\mu=(\mu_1,\mu_2,\ldots,\mu_n)\in L_u$ with $u>0$. Then*

$$|\phi_t^{-1}(\mu)| = \begin{cases} 0 & \text{if } \mu_{t-1}=\mu_t \textit{ and } \mu_t+1\neq\mu_l \textit{ for any } l\in[n], \\ 1 & \text{if } \mu_{t-1}<\mu_t \textit{ and } \mu_t+1\neq\mu_l \textit{ for any } l\in[n] \\ & \text{or } \mu_{t-1}=\mu_t \textit{ and } \mu_t+1=\mu_l \textit{ for some } l\in[n], \\ 2 & \text{if } \mu_{t-1}<\mu_t \textit{ and } \mu_t+1=\mu_l \textit{ for some } l\in[n], \end{cases}$$

where $[n]=\{1,2,\ldots,n\}$. In addition, if $\lambda,v\in\phi_t^{-1}(\mu)$ with $\lambda\not\preccurlyeq v$, then $v\preccurlyeq\lambda$ and $\gamma_{t,t+1}(v)=\lambda$. In this case, $v_t=v_{t+1}=\lambda_t+1$ and $\lambda_t+2=\lambda_l$ for some $l\in[n]$.

Proof. Set $\mu_t=i$. We first consider the case $m_{i+1}(\mu)=0$. If $\lambda\in\phi_t^{-1}(\mu)$, then $\lambda_t\leqslant\mu_t$, and there is an $l\geqslant t$ with $\lambda_t+1=\mu_l$. If $\lambda_t=\mu_t$, then $\lambda_t+1=\mu_l=i+1$ for some $l>t$. This contradicts our assumption $m_{i+1}(\mu)=0$. Therefore, $\lambda_t<\mu_t$. Since $|\lambda|+1=|\mu|$, we have that $\lambda_t=\mu_t-1, \lambda_l=\mu_l$ for $l\neq t$, and $\mu_{t-1}=\lambda_{t-1}<\lambda_t+1=\mu_t$. Thus we see that $\phi_t^{-1}(\mu)=\emptyset$ if $\mu_{t-1}=\mu_t$ and $\phi_t^{-1}(\mu)$ is a singleton if $\mu_{t-1}<\mu_t$.

Next, we suppose that $m_{i+1}(\mu)\neq 0$, more precisely, $\mu_t=\cdots=\mu_{t+r}=i$, and $\mu_{t+r+1}=i+1$. Taking

$$v=(\mu_1,\ldots,\mu_t,\ldots,\mu_{t+r},\mu_{t+r+1}-1,\ldots,\mu_n),$$

we have that $\phi_t(v)=\mu$. In this case, we have that $v_t=v_{t+1}=\mu_t$. Clearly, if $\mu_{t-1}=\mu_t$, then v is the only element of $\phi_t^{-1}(\mu)$. If $\mu_{t-1}<\mu_t$, we may have another element of $\phi_t^{-1}(\mu)$ as follows

$$\lambda=(\mu_1,\ldots,\mu_{t-1},\mu_t-1,\mu_{t+1},\ldots,\mu_{t+r+1},\ldots,\mu_n).$$

It is easy to check that v and λ are the only elements of $\phi_t^{-1}(\mu)$ and they satisfy that $v_t=v_{t+1}=\lambda_t+1$ and $\lambda_t+2=\lambda_{t+r+1}$. $\square$

Now, we fix $n=r+s+1$ and $u=sk+c$, where $0\leqslant c<k$, and r and s are positive integers. For convenience, we write $\phi_{r+1}=\psi$.

Let A be a subset of L_u. We say an element μ of A is maximum (maximal) in $(A,\preccurlyeq)$ if $\lambda\preccurlyeq\mu$ for every $\lambda\in A$ (there is no $v\in A$ with $\mu\prec v$).

For $\mu\in L_{u+1}$, define $d_\mu(p)=f_\mu(p)p^{s-r}-|\psi^{-1}(\mu)|_p$, and for a subset M of L_{u+1}, define $d_M(p)=\sum_{\mu\in M}d_\mu(p)$.

Let L_{u+1}^-, L_{u+1}^0 and L_{u+1}^+ denote the sets of $\mu\in L_{u+1}$ such that $d_\mu(p)<0$, $d_\mu(p)=0$ and $d_\mu(p)>0$, respectively, and let L_u^-, L_u^0 and L_u^+ denote the sets of $\lambda\in L_u$ such that $\psi(\lambda)\in L_{u+1}^-$, $\psi(\lambda)\in L_{u+1}^0$ and $\psi(\lambda)\in L_{u+1}^+$, respectively. Set $L_{u+1}^\pm=L_{u+1}^+\cup L_{u+1}^-$ and $L_u^\pm=L_u^+\cup L_u^-$. Then $|L_{u+1}^0|_p\,p^{s-r}=|L_u^0|_p$ and $d_{L_{u+1}^\pm}(p)=|L_{u+1}^\pm|_p\,p^{s-r}-|L_u^\pm|_p$.

Lemma 8. *Let* $\lambda=(\lambda_1,\ldots,\lambda_n)\in L_u$ *with* $\lambda_{r+1}=i$ *and* $m(\lambda)=(m_0,m_1,\ldots,m_k)$ *such that* λ *is maximum in* $\psi^{-1}(\mu)$, *where* $\mu=\psi(\lambda)$. *Then*

(i) $\lambda\in L_u^-$ *if and only if* $\lambda_r<\lambda_{r+1}<\lambda_{r+2}\leqslant\lambda_{r+1}+2$, *or* $\lambda_r=\lambda_{r+1}<\lambda_{r+2}$ *and* $m_i<m_{i+1}+1$, *or* $\lambda_r=\lambda_{r+1}<\lambda_{r+2}$, $m_i=m_{i+1}+1$ *and* $m_{i+2}>0$;
(ii) $\lambda\in L_u^0$ *if and only if* $\lambda_r<\lambda_{r+1}<\lambda_{r+2}-2$, *or* $\lambda_r=\lambda_{r+1}<\lambda_{r+2}$, $m_i=m_{i+1}+1$ *and* $m_{i+2}=0$;
(iii) $\lambda\in L_u^+$ *if and only if* $\lambda_r=\lambda_{r+1}=\lambda_{r+2}$, *or* $\lambda_r=\lambda_{r+1}<\lambda_{r+2}$ *and* $m_i>m_{i+1}+1$.

Proof. From Lemma 7 it follows that if $\lambda_{r+1}<\lambda_{r+2}$ and $m_{i+2}>0$ then there is another element $v\in\psi^{-1}(\mu)$, where v is given by $m(v)=(m_0,\ldots,m_i-1,m_{i+1}+2,m_{i+2}-1,\ldots,m_k)$. By (11) we have

$$\frac{f_\lambda(p)}{f_v(p)}=\frac{\langle m_{i+1}+2\rangle\langle m_{i+1}+1\rangle}{\langle m_i\rangle\langle m_{i+2}\rangle}\,p^{m_i+m_{i+2}-1},$$

so

$$\deg f_\lambda(p)=\deg f_v(p)+2(m_{i+1}+1). \tag{16}$$

Firstly, we consider the case of $\lambda_r<\lambda_{r+1}=i$. In this case, $\sum_{l<i}m_l=r$ and $\sum_{h>i+1}m_h=s+1-m_i-m_{i+1}$. Then

$$\frac{f_\mu(p)p^{s-r}}{f_\lambda(p)}=\frac{\langle m_i\rangle}{\langle m_{i+1}+1\rangle}\,p^{m_i+m_{i+1}-1}\begin{cases}=\ \frac{p^{m_{i+1}}}{\langle m_{i+1}+1\rangle} & \text{if } m_i=1,\\ >\ p & \text{if } m_i>1.\end{cases} \tag{17}$$

If $m_i>1$, then $\lambda\leqslant\gamma_{r+1,r+2}(\lambda)\in\psi^{-1}(\mu)$, which contradicts our assumption on λ. Therefore, $m_i=1$, and by (17), $f_\mu(p)p^{s-r}=f_\lambda(p)$ if $m_{i+1}=0$. Thus, if $m_{i+1}=m_{i+2}=0$, then $\psi^{-1}(\mu)=\{\lambda\}$ is single, and $d_\mu(p)=f_\mu(p)p^{s-r}-f_\lambda(p)=0$, yielding $\lambda\in L_u^0$. If $m_{i+1}=0$ but $m_{i+2}>0$, then $\psi^{-1}(\mu)=\{\lambda,v\}$ is double, and $d_\mu(p)=f_\mu(p)p^{s-r}-f_\lambda(p)-f_v(p)=-f_v(p)$, yielding $\lambda\in L_u^-$. By (16), we have

$$\deg d_\mu(p)=\deg f_\lambda(p)-2. \tag{18}$$

If $m_{i+1}>0$, then

$$f_\mu(p)p^{s-r}-f_\lambda(p)=\left(\frac{p^{m_{i+1}}}{\langle m_{i+1}+1\rangle}-1\right)f_\lambda(p)=-\frac{\langle m_{i+1}\rangle}{\langle m_{i+1}+1\rangle}\,f_\lambda(p),$$

yielding $\lambda\in L_u^-$ and

$$\deg d_\mu(p)=\deg f_\lambda(p)-1, \tag{19}$$

regardless of whether $\psi^{-1}(\mu)$ is single or double.

Next, we consider the case of $\lambda_r=\lambda_{r+1}=i$ (so $m_i>1$). Suppose $\lambda_r=\lambda_{r+1}=\cdots=\lambda_{r+t}=i<\lambda_{r+t+1}$. Then $\sum_{l<i}m_l=r+t-m_i$ and $\sum_{h>i+1}m_h=s+1-t-m_{i+1}$. Thus,

$$\frac{f_\mu(p)p^{s-r}}{f_\lambda(p)}=\frac{\langle m_i\rangle}{\langle m_{i+1}+1\rangle}\,p^{2t-1+m_{i+1}-m_i}\begin{cases}=\ \frac{\langle m_i\rangle}{\langle m_{i+1}+1\rangle}\,p^{1+m_{i+1}-m_i} & \text{if } t=1,\\ >\ p & \text{if } t>1.\end{cases} \tag{20}$$

From Lemma 7 it follows that $\psi^{-1}(\mu)$ is single if $t>1$. Therefore, $\lambda\in L_u^+$ if $t>1$. Now, suppose $t=1$, i.e., $\lambda_{r+1}<\lambda_{r+2}$. Then $\psi^{-1}(\mu)$ is single if and only if $m_{i+2}=0$. In this case, we have: (a) If $m_{i+1}=0$, then

$$f_\mu(p)p^{s-r}-f_\lambda(p)=\left(\frac{\langle m_i\rangle}{p^{m_i-1}}-1\right)f_\lambda(p)=\frac{\langle m_i-1\rangle}{p^{m_i-1}}f_\lambda(p)>0,$$

so $\lambda\in L_u^+$ and

$$\deg d_\mu(p)=\deg f_\lambda(p)-1, \tag{21}$$

regardless of whether $\psi^{-1}(\mu)$ is single or double. (b) If $m_{i+1}>0$ and $m_i=m_{i+1}+1$, then $f_\mu(p)p^{s-r}=f_\lambda(p)$, so $\lambda\in L_u^0$ if $m_{i+2}=0$, and $\lambda\in L_u^-$ if $m_{i+2}>0$. (c) If $m_{i+1}>0$ and $m_i\neq m_{i+1}+1$, then

$$\begin{aligned}f_\mu(p)p^{s-r}-f_\lambda(p)&=\left(\frac{\langle m_i\rangle\, p^{m_{i+1}+1}}{\langle m_{i+1}+1\rangle\, p^{m_i}}-1\right)f_\lambda(p)\\&=\begin{cases}\dfrac{\langle m_i-m_{i+1}-1\rangle}{\langle m_{i+1}+1\rangle}\,p^{m_{i+1}+1-m_i}f_\lambda(p) & \text{if } m_i>m_{i+1}+1,\\[2ex] -\dfrac{\langle m_{i+1}+1-m_i\rangle}{\langle m_{i+1}+1\rangle}\,f_\lambda(p) & \text{if } m_i<m_{i+1}+1,\end{cases}\end{aligned}$$

which implies that $\deg(f_\mu(p)p^{s-r}-f_\lambda(p))\geqslant\deg f_\lambda(p)-m_{i+1}-1>\deg f_\lambda(p)-2(m_{i+1}+1)$. It follows from (16) that even if there is another element v in $\psi^{-1}(\mu)$, we still have $\lambda\in L_u^+$ if $m_i>m_{i+1}+1$, and $\lambda\in L_u^-$ if $m_i<m_{i+1}+1$. Thus we obtain (i)–(iii). $\square$

For $A\subseteq L_u$ and $\lambda\in L_u$, define $A[\lambda]=\{v\in A\colon\ v\preccurlyeq\lambda\}$, and write

$$\lambda^{(i)}=(\lambda_1,\ldots,\lambda_r,\lambda_{r+1}+i,\lambda_{r+2}-i,\ldots,\lambda_n),\quad i=0,1,\ldots,t_1(\lambda)$$

and

$$\lambda^{[j]}=(\lambda_1,\ldots,\lambda_r+j,\lambda_{r+1}-j,\lambda_{r+2},\ldots,\lambda_n),\quad j=0,1,\ldots,t_2(\lambda),$$

where $t_1(\lambda)=\lfloor\frac{\lambda_{r+2}-\lambda_{r+1}}{2}\rfloor$ and $t_2(\lambda)=\lfloor\frac{\lambda_{r+1}-\lambda_r}{2}\rfloor$. For convenience, we write $\lambda^{(t_1)}=\lambda^{(*)}$ and $\lambda^{[t_2]}=\lambda^{[*]}$. By (14) we have

$$\deg f_{\lambda^{(i)}}(p)=\deg f_{\lambda^{(i+1)}}(p)+2,\quad i=0,1,\ldots,t_1(\lambda)-1,$$

which implies

$$\deg f_\lambda(p)=\deg f_{\lambda^{(*)}}(p)+2t_1(\lambda). \tag{22}$$

Similarly,

$$\deg f_\lambda(p)=\deg f_{\lambda^{[*]}}(p)+2t_2(\lambda). \tag{23}$$

Write $\mu=\psi(\lambda)$, $\mu^{(*)}=\psi(\lambda^{(*)})$ and $\mu^{[*]}=\phi_r(\lambda^{[*]})$. Clearly, if $m_k(\lambda)=s$, then $\nabla(L_u[\lambda])\subseteq L_{u+1}[\mu]$. Set $L_{u+1}^{\pm}[\mu]=L_{u+1}^{\pm}\cap L_{u+1}[\mu]$.

Lemma 9. *Let* $\lambda=(\lambda_1,\ldots,\lambda_n)\in L_u$ *with* $m(\lambda)=(m_0,m_1,\ldots,m_k)$ *and* $m_k=s$. *Then* $\mu^{(*)}$ *and* $\mu^{[*]}$ *are maximum in* $(L^-_{u+1}[\mu],\preccurlyeq)$ *and* $(L^+_{u+1}[\mu],\preccurlyeq)$, *respectively, and*

$$\deg d_{L^-_{u+1}[\mu]}(p)=\deg d_{\mu^{(*)}}(p)=\deg f_\lambda(p)-(\lambda_{r+2}-\lambda_{r+1}) \tag{24}$$

and

$$\deg d_{L^+_{u+1}[\mu]}(p)=\deg d_{\mu^{[*]}}(p)=\deg f_\lambda(p)-(\lambda_{r+1}-\lambda_r)-1. \tag{25}$$

Proof. Let π be a maximal element of $(L^-_{u+1}[\mu],\preccurlyeq)$ and let σ be the maximum element of $(\psi^{-1}(\pi),\preccurlyeq)$. Then it is easy to see that σ is maximal in $(L^-_u[\lambda],\preccurlyeq)$. If $\sigma_r=\sigma_{r+1}=i$, then, by (i) in Lemma 8, $\sigma_{r+2}=i+1$ and $m_i(\sigma)\leqslant m_{i+1}(\sigma)+1$. In this case, from $k>1$ and $c<k$ it follows $\sigma_{r+2}<k=\lambda_{r+2}$. Therefore, $\sigma_1+\cdots+\sigma_{r+1}>\lambda_1+\ldots+\lambda_{r+1}$. From (8) it is easy to verify that $\delta=\gamma_{r,r+2}(\sigma)\preccurlyeq\lambda$ and, by (i) in Lemma 8 again, $\delta\in L^-_u$, hence $\delta\in L^-_u[\lambda]$, a contradiction, which yields $\sigma_r<\sigma_{r+1}$, and so $\sigma_{r+1}<\sigma_{r+2}\leqslant i+2$. We first consider the case of $\sigma_{r+2}=i+1$. Under this condition, if $\sigma_1+\cdots+\sigma_r>\lambda_1+\cdots+\lambda_r$, then $\sigma_{r+2}<k$, and we still have $\delta\in L^-_u[\lambda]$, a contradiction again. The contradiction shows $\sigma_1+\cdots+\sigma_r=\lambda_1+\cdots+\lambda_r$, from which and that σ is maximal in $L^-_u[\lambda]$ it follows $(\sigma_1,\ldots,\sigma_r)=(\lambda_1,\ldots,\lambda_r)$. In this case, if $n>r+2$ but $\sigma_{r+3}<k$, then $\gamma_{r+1,r+3}(\sigma)\in L^-_u[\lambda]$. The contradiction shows that $\sigma_{r+3}=k$ (if $n>r+2$) and $\sigma_{r+1}+\sigma_{r+2}=\lambda_{r+1}+\lambda_{r+2}=2i+1$. So $\lambda_{r+2}-\lambda_{r+1}=2t_1(\lambda)+1$, $\sigma_{r+1}=\lambda_{r+1}+t_1(\lambda)$ and $\sigma_{r+2}=\lambda_{r+2}-t_1(\lambda)=\lambda_{r+1}+t_1(\lambda)+1$, that is, $\sigma=\lambda^{(*)}$ and $\pi=\mu^{(*)}$. Thus we have that $\mu^{(*)}$ is maximum in $L^-_{u+1}[\mu]$. By (19) and (22) we have

$$\begin{aligned}\deg d_{\mu^{(*)}}(p)&=\deg f_{\lambda^{(*)}}(p)-1=\deg f_\lambda(p)-2t_1(\lambda)-1\\&=\deg f_\lambda(p)-(\lambda_{r+2}-\lambda_{r+1}).\end{aligned}$$

Now, we consider the case of $\sigma_{r+2}=i+2$. Under this condition, we have $\gamma_{r,r+1}(\sigma)\in L^-_u[\lambda]$ if $(\sigma_1+\cdots+\sigma_r)>(\lambda_1+\cdots+\lambda_r)$, and $\gamma_{r+2,r+3}(\sigma)\in L^-_u[\lambda]$ if $(\sigma_1,\ldots,\sigma_r)=(\lambda_1,\ldots,\lambda_r)$ but $n>r+2$ and $\sigma_{r+3}<k$, which implies $\sigma_{r+1}+\sigma_{r+2}=\lambda_{r+2}+\lambda_{r+1}=2i+2$. So $\lambda_{r+2}-\lambda_{r+1}=2t_1(\lambda)$, $\sigma_{r+1}=\lambda_{r+1}+t_1(\lambda)-1$ and $\sigma_{r+2}=\lambda_{r+2}-t_1(\lambda)+1=\lambda_{r+1}+t_1(\lambda)+1$, that is, $\sigma=\lambda^{(t_1(\lambda)-1)}$ and $\pi=\mu^{(*)}$. By (18) we have

$$\deg d_{\mu^{(*)}}(p)=\deg f_{\lambda^{(*)}}(p)=\deg f_\lambda(p)-2t_1(\lambda)=\deg f_\lambda(p)-(\lambda_{r+2}-\lambda_{r+1}).$$

For any $\pi\in L^-_{u+1}[\mu]$, if $\pi\neq\mu^{(*)}$, then $\pi\prec\mu^{(*)}$, by (15), $\deg f_\pi(p)\leqslant\deg f_{\mu^{(*)}}(p)-2$, so $\deg d_\pi(p)<\deg d_{\mu^{(*)}}(p)$, which leads to (24).

In a similar way, we can prove: (1) If $\lambda_{r+1}-\lambda_r=2t_2(\lambda)$, then $\lambda_r+t_2(\lambda)=\lambda_{r+1}-t_2(\lambda)$ and $\lambda^{[*]}=(\lambda_1,\ldots,\lambda_{r-1},\lambda_r+t_2(\lambda),\lambda_{r+1}-t_2(\lambda),k^s)$ is maximum in $(L^+_u[\lambda],\preccurlyeq)$ and $\mu^{[*]}=\phi_r(\lambda^{[*]})=\psi(\lambda^{[*]})$ is maximum in $(L^+_{u+1}[\mu],\preccurlyeq)$. In this case, $\deg d_{L^+_{u+1}[\mu]}(p)=\deg d_{\mu^{[*]}}(p)=\deg f_{\lambda^{[*]}}(p)-1=\deg f_\lambda(p)-2t_2(\lambda)-1=\deg f_\lambda(p)-(\lambda_{r+1}-\lambda_r)-1$. (2) If $\lambda_{r+1}-\lambda_r=2t_2(\lambda)+1$, then $\lambda_r+t_2(\lambda)+1=\lambda_{r+1}-t_2(\lambda)$ and $\lambda^{[*]}=(\lambda_1,\ldots,\lambda_{r-1},\lambda_r+t_2(\lambda),\lambda_{r+1}-t_2(\lambda),k^s)$ is maximum in $(L^+_u[\lambda],\preccurlyeq)$ and $\mu^{[*]}=\phi_r(\lambda^{[*]})$ is maximum in $(L^+_{u+1}[\mu],\preccurlyeq)$. In this case, $\deg d_{L^+_{u+1}[\mu]}(p)=\deg d_{\mu^{[*]}}(p)=\deg f_{\mu^{[*]}}(p)+s-r$. By (10), $f_{\mu^{[*]}}(p)p^{s-r}/f_{\lambda^{[*]}}(p)=\langle m_i(\lambda^{[*]})\rangle/\langle 2\rangle\, p^{m_i(\lambda^{[*]})}$, so $\deg d_{L^+_{u+1}[\mu]}(p)=\deg f_{\lambda^{[*]}}(p)-2=\deg f_\lambda(p)-2t_2(\lambda)-2=\deg f_\lambda(p)-(\lambda_{r+1}-\lambda_r)-1$. In both cases (25) follows. $\square$

4. Proof of Theorem 2

For any $\lambda \in L(n,k)$, define $\lambda'' = (k^n) - \lambda$, i.e., $\lambda''_i = k - \lambda_{n-i+1}$, $i = 1,2,\dots,n$, or $m_j(\lambda'') = m_{k-j}(\lambda)$, $j = 0,1,\dots,k$. Then $\lambda \to \lambda''$ is an order-reversing map of L. In other words, for any λ, $\mu \in L$, $\lambda \leqslant \mu$ implies $\mu'' \leqslant \lambda''$. We call λ'' the symmetry of λ. It is well known that $f_\lambda(p) = f_{\lambda''}(p)$ (see e.g. [17, p. 188]). Set $B'' = \{\lambda'' : \lambda \in B\}$. Then $|B''|_p = |B|_p$.

Suppose $n = r + s + 1 \geqslant 2$, $k \geqslant 2$ and $u = sk + c$, where $0 \leqslant c < k$. Let A be a subset of L_u, $B = \nabla(A) = \{\mu \in L_{u+1} : \mu \geqslant \lambda \text{ for some } \lambda \in A\}$, $\bar{A} = L_u \backslash A$ and $\bar{B} = L_{u+1} \backslash B$. It is easy to verify that $\bar{A}'' \supseteq \nabla(\bar{B}'')$, and $|L_{u+1}|_p = |B|_p + |\bar{B}|_p = |B|_p + |\bar{B}''|_p = |L_{nk-u-1}|_p$ and $|L_u|_p = |A|_p + |\bar{A}|_p = |A|_p + |\bar{A}''|_p = |L_{nk-u}|_p$. Thus we immediately have

$$\frac{|B|_p}{|A|_p} \geqslant \frac{|L_{u+1}|_p}{|L_u|_p} \quad \Leftrightarrow \quad \frac{|\bar{A}''|_p}{|\bar{B}''|_p} \geqslant \frac{|L_{nk-u}|_p}{|L_{nk-u-1}|_p}. \tag{26}$$

Let $\alpha = (0^r, c, k^s)$ and $\beta = (0^r, c+1, k^s)$ be the maximum elements of $(L_u, \preccurlyeq)$ and $(L_{u+1}, \preccurlyeq)$, respectively. Then $\beta = \phi(\alpha) = \psi(\alpha)$, and $L_u[\alpha] = L_u$ and $L_{u+1}[\beta] = L_{u+1}$. By (13), $\deg f_\beta(p) - \deg f_\alpha(p) = n - 1 - 2m_k(\alpha) = r - s = \deg |L_{u+1}|_p - \deg |L_u|_p$.

If $\alpha \in A$, then $\beta \in B$, which implies that the maximum element β'' of L_{nk-u-1} does not belong to $\bar{B}''$. Without loss of generality, we assume $\alpha \notin A$. (If not, we can take $\bar{B}''$ as the A.)

Suppose $\lambda \in A$ such that $\deg |A|_p = \deg f_\lambda(p)$. Then $\phi(\lambda) \in B$ and

$$\deg \frac{|B|_p}{|A|_p} \geqslant \deg \frac{f_{\phi(\lambda)}(p)}{f_\lambda(p)} = n - 1 - 2m_k(\lambda) = r + s - 2m_k(\lambda).$$

If $m_k(\lambda) < s$, then $\deg |L_{u+1}|_p / |L_u|_p < \deg |B|_p / |A|_p$, which completes our proof. If $r = 0$ or $c \leqslant 1$, then $m_k(\lambda) < s$ since $\lambda \neq \alpha = (0^r, c, k^s)$.

Now, we assume $r > 0$, $c > 1$ and $m_k(\lambda) = s$.

For any $\mu \in B$, define $\bar{d}_\mu(p) = f_\mu(p)p^{s-r} - |\psi^{-1}(\mu) \cap A|_p$, and for a subset N of B, define $\bar{d}_N(p) = \sum_{\mu \in N} \bar{d}_\mu(p)$. Then A^0, A^-, A^+, $A^\pm$, B^0, B^-, B^+ and $B^\pm$ are defined analogously to L^0, L^-, etc. From definition we see that $|B^0|_p p^{s-r} = |A^0|_p$, $\bar{d}_{B^\pm}(p) = |B^\pm|_p p^{s-r} - |A^\pm|_p$ and $\deg \bar{d}_N(p) = \deg d_N(p)$ for $N \subseteq B^-$.

It is easy to see that

$$\begin{aligned}
&\frac{|B|_p}{|A|_p} \geqslant \frac{|L_{u+1}|_p}{|L_u|_p} \\
&\quad \Leftrightarrow p^{s-r}(|B^0|_p + |B^\pm|_p)(|L_u^0|_p + |L_u^\pm|_p) \\
&\quad \geqslant p^{s-r}(|A^0|_p + |A^\pm|_p)(|L_{u+1}^0|_p + |L_{u+1}^\pm|_p) \\
&\quad \Leftrightarrow p^{s-r}|B^0|_p|L_u^\pm|_p + p^{s-r}|B^\pm|_p|L_u^0|_p + p^{s-r}|B^\pm|_p|L_u^\pm|_p \\
&\quad \geqslant p^{s-r}|A^0|_p|L_{u+1}^\pm|_p + p^{s-r}|A^\pm|_p|L_{u+1}^0|_p + p^{s-r}|A^\pm|_p|L_{u+1}^\pm|_p \\
&\quad \Leftrightarrow |A^0|_p|L_u^\pm|_p + (\bar{d}_{B^\pm}(p) + |A^\pm|_p)|L_u^0|_p + (\bar{d}_{B^\pm}(p) + |A^\pm|_p)|L_u^\pm|_p \\
&\quad \geqslant |A^0|_p(d_{L_{u+1}^\pm}(p) + |L_u^\pm|_p) + |A^\pm|_p|L_u^0|_p + |A^\pm|_p(d_{L_{u+1}^\pm}(p) + |L_u^\pm|_p) \\
&\quad \Leftrightarrow \bar{d}_{B^\pm}(p)|L_u|_p \geqslant |A|_p d_{L_{u+1}^\pm}(p),
\end{aligned} \tag{27}$$

$$\Leftrightarrow |L_u|_p(\bar{d}_{B^+}(p)+\bar{d}_{B^-}(p)) \geqslant |A|_p(d_{L_{u+1}^+}(p)+d_{L_{u+1}^-}(p)). \tag{28}$$

We first prove

$$|L_u|_p\bar{d}_{B^+}(p) \geqslant |A|_p d_{L_{u+1}^+}(p). \tag{29}$$

Recall that $\deg|A|_p = \deg f_\lambda(p)$, where $\lambda=(\lambda_1,\ldots,\lambda_{r+1},k^s)\neq\alpha$. Then $\lambda_r>0$ and $\lambda_{r+1}<c$. If $\lambda^{[*]}\in A$, then $\mu^{[*]}=\phi_r(\lambda^{[*]})\in B^+$ and $\deg\bar{d}_{B^+}(p)\geqslant\deg d_{\mu^{[*]}}(p)$. By Lemma 9, $\deg d_{\mu^{[*]}}(p)=\deg f_\lambda(p)-(\lambda_{r+1}-\lambda_r)-1$ and $\deg d_{L_{u+1}^+}(p)=\deg f_\alpha(p)-c-1$. Therefore,

$$\begin{aligned}\deg(|L_u|_p\bar{d}_{B^+}(p)) &\geqslant \deg f_\alpha(p)+\deg f_\lambda(p)-(\lambda_{r+1}-\lambda_r)-1\\ &> \deg|A|_p+\deg f_\alpha(p)-c-1=\deg(|A|_p d_{L_{u+1}^+}(p)).\end{aligned} \tag{30}$$

If $\lambda^{[*]}\notin A$, then there is an index j_0, $0\leqslant j_0<t_2(\lambda)$, such that $\lambda^{[j_0]}\in A$ but $\lambda^{[j_0+1]}\notin A$. From $\psi(\lambda^{[j_0+1]})=\mu^{[j_0+1]}=(\lambda_1,\ldots,\lambda_r+j_0+1,\lambda_{r+1}-j_0,k^s)\geqslant(\lambda_1,\ldots,\lambda_r+j_0,\lambda_{r+1}-j_0,k^s)=\lambda^{[j_0]}$ it follows that $\mu^{[j_0+1]}\in B$ and $\bar{d}_{\mu^{[j_0+1]}}(p)=f_{\mu^{[j_0+1]}}(p)p^{s-r}>d_{\mu^{[*]}}(p)$, which implies (30) again. Since $|L_u|_p\bar{d}_{B^+}(p)$ is positive (29) follows from (30).

If $\deg\bar{d}_{B^-}(p)<\deg\bar{d}_{B^+}(p)$, then (28) follows from (10). It is not difficult to see from (13) that $B^-=\emptyset$ if $s=0$. We now assume $s>0$ and $\deg\bar{d}_{B^-}(p)\geqslant\deg\bar{d}_{B^+}(p)$ and proceed to prove that

$$|L_u|_p\bar{d}_{B^-}(p)\geqslant|A|_p d_{L_{u+1}^-}(p). \tag{31}$$

Let $\sigma\in A^-$ such that $\deg\bar{d}_\pi(p)=\deg\bar{d}_{B^-}(p)$, where $\pi=\psi(\sigma)$. Then $\sigma_{r+1}<\sigma_{r+2}$. Set $\pi^{\{1\}}=\phi_{r+2}(\sigma)$ if $\sigma_{r+2}<k$. Then $\pi^{\{1\}}\in B$. If $\psi^{-1}(\pi^{\{1\}})\cap A\neq\emptyset$, we let $\sigma^{\{1\}}$ denote the maximum element of it and set $\pi^{\{2\}}=\phi_{r+2}(\sigma^{\{1\}})$ if $\sigma_{r+2}^{\{1\}}<k$. Repeat this process until we have $\sigma^{\{t\}}\in A$ such that $\psi^{-1}(\pi^{\{t+1\}})\cap A=\emptyset$ or $\sigma_{r+2}^{\{t\}}=k$. In the former case, we have $\deg\bar{d}_{B^-}(p)<\deg\bar{d}_{B^+}(p)$, which contradicts our assumption. In the latter case, we write $\sigma^{\{t\}}=\delta=(\delta_1,\ldots,\delta_{r+1},k^s)$. Then $\sigma\in L_u[\delta]$. Lemma 9 implies that

$$\deg d_\pi(p)\leqslant\deg f_\delta(p)-(\delta_{r+2}-\delta_{r+1})=\deg f_\delta(p)-k+\delta_{r+1}.$$

Since $\delta\neq\alpha$, $\delta_r>0$ and $\delta_{r+1}<c$. So

$$\deg\bar{d}_{B^-}(p)=\deg d_\pi(p)\leqslant\deg f_\delta(p)-k+\delta_{r+1}<\deg|A|_p-k+c,$$

which implies that

$$\begin{aligned}\deg(|L_u|_p\bar{d}_{B^-}(p)) &< \deg f_\alpha(p)+\deg|A|_p-k+c\\ &= \deg(|A|_p d_{L_{u+1}^-}(p)).\end{aligned} \tag{32}$$

Since $|A|_p d_{L_{u+1}^-}(p)$ is negative (31) follows from (32).

The proof is complete. □

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